Notes on Z_N dyons

Thanks for your comments. I will refer first to the mathematics which we have read, and then secondly to the physics argument.

In terms of mathematics, I think the relevant reference is the article by S. Sedlacek in Comm. Math. Phys. 86, 515 (1982). In particular, we want to consider bundles with nontrivial obstruction. The obstruction, η, is defined using Cech cohomology. It is an element in $H^2(M, \pi_1(G/H))$. This is the *second* cohomology group, which in your analysis didn't enter at all. For a pure gauge theory, with $G/H = SU(N)/Z(N)$, $\pi_1(G/H) = Z(N)$.

Sedlacek refers to the obstruction η as being a condition which Taubes needs to vanish, and which is related to 't Hooft's "twist".

In his theorem 2.4, Sedlacek classifies eta for $G = SO(N)$ and $U(N)$. He does *not* do $SU(N)$. (In the second to last sentence of his paper, in the Appendix, Sedlacek does refer to $\eta = H^2(M, Z(N))$, but we don't understand the rest of the discussion.)

So let us consider $SO(3)$. Then η is equal to the second Stiefel-Whitney class of the vector bundle.

The same cohomology group arises in the discussion of Avis and Isham which I faxed you. In their eq. (4.22), they have the classifying exact sequence:

$$\rightarrow\ H^4(M; \pi_3(G/H))\ \rightarrow\ \mathcal{B}_{\mathcal{G}}(M)\ \rightarrow\ H^2(M; \pi_1(G/H))\ \rightarrow\ 0. \qquad (1)$$

While there is the H^4 on the left, there is also a H^2, identical to that of Sedlacek, on the right. Avis and Isham also consider the case of $SO(3)$, and in (4.64) derive the "well-known" constraint that the Pontryagin number, mod 2, equals the square of the second Stiefel-Whitney class. By eq. (4.81), Avis and Isham mention the case of $SU(3)/Z(3)$, but they claim the embedding is into $SO(8)$. This doesn't sound right to me.

What about physics? Well it is known, from both 't Hooft and from van Baal, that in a finite box, that the second Chern class is not integral, as for an instanton, but comes in fractions of $1/N$.

Now you say this is special to a box. I demur. Consider a box, with all three sides equal. I can construct a configuration in which there is a unit of topological charge $= 1/N$. This will be asymmetric, with the flux running through the x direction, say.

But why don't I just take radial boundary conditions? That is, I construct a configuration with nontrivial A_0 and A_i at infinity. The boundary conditions can be read off by requring that the configuration carry nontrivial radial $Z(N)$ magnetic charge, and that it carry a twist in A_0.

For A_0, at $r = \infty$ we choose

$$A_0\ =\ \frac{2\pi T}{N}\,\mathbf{k}\,. \qquad (2)$$

Here $\mathbf{k}$ is a diagonal $SU(N)$ matrix related to $Z(N)$ transformations. I choose elements of $\vec{k}$ to be integral, for convenience. Then the only two choices are

$$\mathbf{c_1} \;=\; \begin{pmatrix} \mathbf{1}_{N-1} & 0 \\ 0 & -(N-1) \end{pmatrix} \quad , \quad \mathbf{c_2} \;=\; \begin{pmatrix} \mathbf{1}_{N-2} & 0 & 0 \\ 0 & -(N-1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

For $\mathbf{k}$ equal to either $\mathbf{c_i}$, it is clear that the Wilson line in the imaginary time direction,

$$\Omega \;=\; \exp\left(i \int_0^{1/T} A_0 \, d\tau \right) \;=\; \exp\left(\frac{2\pi i}{N}\, \mathbf{k} \right) \;, \tag{4}$$

gives a nontrivial twist. This is also called holonomy, I believe.

The boundary conditions for the spatial components are *almost* those of the Dirac monopole. We divide a sphere into an upper and a lower hemisphere, with gauge potentials on each, $A^{\pm}$. Then

$$A^{\pm}_{\phi} \;=\; \frac{1}{2Nr}\, \mathbf{m}\, \frac{(\pm\mathbf{1} - \cos\theta)}{\sin\theta} \;. \tag{5}$$

To see this is a $Z(N)$ monopole, compute the the Wilson line for a special closed path, $\vec{s}$. With θ and ϕ polar angles, $\hat{z} = \cos\theta$, take a path at constant θ, wrapping around by 2π in ϕ, so that $\vec{A}\cdot d\vec{s} = A_\phi r d\phi$. This can be done at any $\theta \neq 0, \pi$. Since the vector potential is specified by two patches, we compute the following quantity. First, take the Wilson line with A^+, going around by 2π in ϕ; then, take the Wilson line with A^-, running in the opposite direction:

$$\exp\left(i \oint \vec{A}^+ \cdot d\vec{s} \right) \left(\exp\left(i \oint \vec{A}^- \cdot d\vec{s} \right) \right)^{\dagger} \;=\; \exp\left(\frac{2\pi i}{N}\, \mathbf{m} \right) \;. \tag{6}$$

This is manifestly gauge invariant, and $= 1$ if the configuration is trivial, $A^+ = A^-$. For the $Z(N)$ monoopole, instead one obtains a non-trivial element of $Z(N)$. For this to be true, $\mathbf{m}$ must be one of the two matrices, $\mathbf{c_1}$ or $\mathbf{c_2}$.

The above is basically a construction of a $Z(N)$ Wu-Yang monopole. Notice that it has nothing to do with $\pi_2(G/H)$, only $\pi_1(G/H)$.

The above are the boundary conditions at spatial infinity, $r \to \infty$. At the origin, $r = 0$, we require all A_μ's to vanish, at least like $\sim r^2$, so that $G_{\mu\nu} \sim r$ as $r \to 0$. We do not know what the general configuration looks like. Surely it is self-dual. In that case, for large r,

$$A_0(r) \;=\; \frac{2\pi T}{N}\, \mathbf{k} \;-\; \frac{\mathbf{1}}{\mathbf{2Nr}}\, \mathbf{m} \;+\; \ldots \tag{7}$$

We assume that the configuration is static. In that case, it is easy to compute the topological charge:

$$Q \;=\; \frac{1}{4\pi^2} \int d^4x \; \partial_i \; \mathrm{tr}\,(A_0 \; B_i) \;. \tag{8}$$

There are of course other terms in the topological charge, but we drop them, as they shouldn't contribute for a static configuration. In that case, using the above it is easy to compute the topological charge, and find

$$Q \;=\; \frac{1}{N^2}\,\mathbf{m}\cdot\mathbf{k}\;. \tag{9}$$

A similar equation was derived by 't Hooft in his Schladming lectures. We differ by a factor of $1/N$, but he didn't carefully define the normalization of his charges.

There are only two cases to consider. Either the $Z(N)$ charges are the same, or they are different. If they are the same, $\mathbf{m} = \mathbf{k} = \mathbf{c_1}$,

$$Q \;=\; \frac{N-1}{N}\;. \tag{10}$$

If they charges are different, such as $\mathbf{m} = \mathbf{c_1}$ and $\mathbf{k} = \mathbf{c_2}$, then

$$Q \;=\; -\,\frac{1}{N}\;. \tag{11}$$

You assert that there should be no such solution which satisfies the Yang-Mills equations of motion with the above boundary conditions. We don't see why not. The existence of nontrivial $Z(N)$ charges would seem to guarantee that you can't get rid of such a knot, having formed it. Note that it is crucial to have both electric and magnetic $Z(N)$ charges. The only scale possible for the configuration is the temperature, T, but that is fine.

The configuration is stable under dilatations. Under $\vec{r} \to \lambda \vec{r}$, let $A_i \to A_i/\lambda$. In contrast, A_0 does *not* scale under dilatations, $A_0 \to A_0$, since its scale is fixed to the temperature. Then the action scales as

$$\int d^3x \;\mathrm{tr}\left(\vec{E}^2 \;+\vec{B}^2\right) \;\to\; \lambda \int d^3x \;\mathrm{tr}\left(\vec{E}^2\right) \;+\; \frac{1}{\lambda}\int d^3x \;\mathrm{tr}\left(\vec{B}^2\right)\;. \tag{12}$$

I only integrate over the three spatial directions, since the configuration is assumed to be static. Requiring that this is stationary with respect to λ just fixes $\lambda = 1$; *i.e.*, the configuration is self-dual, $\vec{E} = \pm\vec{B}$.

If such a configuration exists, then it is self-dual only over distances $\sim 1/T$. Over larger distances, it is not self-dual. This is because static electric fields are screened at one loop order, but static magnetic fields are not. Using the Debye mass $m_D^2 = N(gT)^2/3$, one finds a correction $\sim 1/g$ to the action:

$$\frac{8\pi^2}{g^2 N}\left(-1 \;+\; \frac{(N-1)}{8\pi}\sqrt{\frac{g^2\,N}{3}}\;\right)\;. \tag{13}$$

The computation is simple, but takes more to explain, so I just give the answer. This shows that if instantons are made up of $Z(N)$ dyons, then the dyons have relatively strong interactions; they are not $\sim g^0$, as for instantons, but $\sim 1/g$. This probably means that a dilute gas approximation is not so good for dyons, although the numerical value of the number on the right hand side is amusingly small for a momentum scale of order 1 GeV.

What does worry me is that the corrections to the action are of order N at large N, keeping $g^2 N$ fixed as $N \to \infty$.